Math 3235 Probability Theory

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$X_i \quad i = 1 \ldots N$

i.i.d.

$$\bar{X}_N = \frac{1}{N} \sum_{i=1}^{N} X_i$$

$$\bar{X}_N \xrightarrow{P} \mathbb{E}(X) = \mu$$

$$\lim_{N \to \infty} \mathbb{P}\left(|\bar{X}_N - \mu| > \delta \right) = 0 \qquad \forall \delta$$

$$\mathbb{E}(\bar{X}_N) = \mu$$

$$\mathbb{P}\left(|\bar{X}_N - \mu| > \delta \right) \leq \frac{Var(\bar{X}_N)}{\delta^2}$$

$$Var(\bar{X}_N) = \mathbb{E}\left(\left(\frac{1}{N} \sum_i (X_i - \mu) \right)^2 \right) =$$

$$\left(\sum_i a_i \right)^2 = \sum_{i,j} a_i a_j$$

$$\to \frac{1}{N^2}\mathbb{E}\left(\left(\sum_i (X_i-\mu)\right)^2\right) =$$

Use the formula with

$$a_i = X_i - \mu$$

$$= \frac{1}{N^2}\sum_{i,j}\mathbb{E}\left((X_i-\mu)(X_j-\mu)\right) =$$

if $i \neq j$

$$\mathbb{E}\left((X_i-\mu)(X_j-\mu)\right) =$$

$$\mathbb{E}(X_i-\mu)\,\mathbb{E}(X_j-\mu) = 0$$

$$\mathbb{E}(X_iX_j) = \mathbb{E}(X_i)\mathbb{E}(X_j)$$

$$\mathbb{E}(X_i-\mu) = 0$$

$$= \frac{1}{N^2}\sum_i Var(X_i)$$

$$Var(\bar{X}) = \frac{Var(X_i)}{N}$$

Thus

$$\mathbb{P}\left(|\overline{X}_N - \mu| > \delta\right) \leq \frac{Var(X_1)}{N\delta^2}$$

All we used is that

$$cov(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j) = 0$$

σ

$$\mathbb{P}\left(|X - \mu| > \delta\right) \leq \frac{\sigma^2}{\delta^2}$$

$\mu = \mathbb{E}(X) \qquad \sigma^2 = Var(X)$

$\delta = \sigma t$

$$\mathbb{P}\left(|X - \mu| > \sigma t\right) \leq \frac{\sigma^2}{\sigma^2 t^2} = \frac{1}{t^2}$$

$$M_X(t) = \mathbb{E}(e^{tX})$$

$$M_{X+Y}(t) = M_X(t)\,M_Y(t) \qquad X \perp\!\!\!\perp Y$$

$$M_X(0) = \mathbb{E}(1) = 1$$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$X_1 \quad \dots \quad X_N$$

$$x_1 \qquad x_N$$

0, 1, 0, 1, 1, 0, 1

$$\bar{X}_N = \frac{1}{N}\sum_{i=1}^{N} X_i \xrightarrow{P} p$$

$\hat{p} = \bar{X}_N$ estimator

I want N to be large enough such that

$$\mathbb{P}(|\hat{p} - p| \leq 0.005) = 0.95$$

$$\mathbb{P}\left(\left|\frac{\sum_i X_i}{N} - p\right| \leq 0.005\right)$$

if True prob is p

$$\mathbb{E}(X_i) = p \qquad Var(X_i) = p(1-p)$$

$$\frac{\sum_i X_i - Np}{\sqrt{N}\sqrt{p(1-p)}} \simeq N(0,1)$$

$$\mathbb{P}\left(-0.005 \leq \frac{\sum_i X_i - Np}{N} \leq 0.005\right) =$$

$$\mathbb{P}\left(\frac{-0.005\sqrt{N}}{\sqrt{p(1-p)}} \leq \frac{\sum_i X_i - Np}{\sqrt{N}\sqrt{p(1-p)}} \leq \frac{0.005\sqrt{N}}{\sqrt{p(1-p)}}\right)$$

Find N such That

$$\mathbb{P}\left(\frac{-0.005\sqrt{N}}{\sqrt{p(1-p)}} \le Z \le \frac{0.005\sqrt{N}}{\sqrt{p(1-p)}}\right) > 0.95$$

$$1 - 2\,\Phi\left(-\frac{0.005\sqrt{N}}{\sqrt{p(1-p)}}\right) > 0.95$$

N_k is geometric r.v. with

par $\frac{\lambda}{k}$

$$\mathbb{P}(N_k = n) = (1-p_k)^{n-1} p_k$$

$$Z_k = \frac{N_k}{k} \xrightarrow{d} Y$$

where Y is exponential with

par λ.

$$\mathbb{P}(Z_k \le x) = \mathbb{P}(N_k \le kx) =$$

$$\sum_{i=1}^{\lfloor kx \rfloor} (1-p_k)^{i-1} p_k = 1 - (1-p_k)^{\lfloor kx \rfloor - 1} =$$

$\lfloor x \rfloor$ = floor of x

largest integer smaller than x.

$$= 1 - \left(1 - \frac{\lambda}{k}\right)^{\lfloor kx \rfloor} \xrightarrow{k \to \infty} 1 - e^{-\lambda x} \blacksquare$$

Q.E.D.